Fibre bundle varieties and the number of generations of elementary particles

1985 J. Phys. A: Math. Gen. 18 2457

(http://iopscience.iop.org/0305-4470/18/13/022)

J. Phys. A: Math. Gen. 18 (1985) 2457-2464. Printed in Great Britain

Fibre bundle varieties and the number of generations of elementary particles

D K Ross

Physics Department, Iowa State University, Ames, Iowa 50011, USA

Received 19 February 1985

Abstract. We present the idea that the number of generations of elementary particles in a gauge theory characterised by a given Lie algebra is the same as the number of topologically distinct principal fibre bundles with a structure group having the same Lie algebra and $R^3-\{0\}$ as base space. Two different generations thus have a different global structure or 'twist' to their fibre bundles. We find that at most three generations are allowed for groups with the same Lie algebra as E_6, at most four generations for groups with the same Lie algebra as SO_{4l+2} with $l \geq 2$, and at most n generations for groups with the same Lie algebra as SU_n.

1. Introduction

One of the mysteries of elementary particle physics is why we have three or more generations of particles. Thus in the $SU_2 \times U_1$ unified gauge theory of the weak and electromagnetic interactions (Weinberg 1980, Salam 1980, Glashow 1980), the least massive leptons and quarks are put into the theory as the left-handed doublets $\binom{e^-}{\nu_e}$ and $\binom{u}{d}$ with this pattern repeated twice more for the μ^- and τ^- generations. In SU_5 GUT theory, the number of leptons and quarks is directly linked together. The number of generations or families remains undetermined, however. Thus to have N generations, we need N numbers of $(\mathbf{5}^*+\mathbf{10})$ representations (Tye 1982). Going beyond usual GUT theories can fix N. Thus recent work on a supersymmetric preon model by Greenberg *et al* (1983) gives $N=3$. Another possibility is to go to a gauge group such as SO_{4n+2} or SU_n sufficiently large to accommodate the family replications as well as the usual flavour symmetries. SO_{4n+2} can accommodate 2^{2n-5} families so that SO_{18} can handle 3 families; SU_7 can also handle 3 families (Ross 1984). One can also consider models with a family symmetry with the same group as the flavour symmetry with a discrete symmetry between the factors. $SU_5 \times SU_5$ is one example (Ross 1984). All of these large group models seem rather contrived and do not really elucidate the basic differences between the flavour and the family symmetries, i.e. that the family structure just seems to be xerox copies of the flavour structure with no gauge dynamics associated with the family structure at all. Treating the family structure as a gauge theory seems unjustified and unnecessary.

Is there any natural way that a gauge theory can supply Xerox copies of itself? It has been known for some time that the appropriate mathematical description of a gauge theory is in terms of fibre bundles. Although locally the fibre bundle looks like the direct product of the base space (such as spacetime) and the structure group (gauge group), globally the fibre bundle can be twisted. Thus a given gauge theory with a given gauge group can have associated with it more than one inequivalent fibre bundles. These inequivalent fibre bundles then very naturally supply xerox copies of the gauge theory. Consider the e^-, μ^-, and τ^- families. In this view they would all be described by the same gauge theory with the same gauge connections and gauge fields and hence identical interactions. The difference between these families would lie in the different global twist in their associated fibre bundles. This picture seems very natural and simple in the sense that we do not put in anything more than the gauge theory required for the flavour group. The family structure comes out automatically and is not gauged in any way. The fact that the different families have identical interactions and yet at least three xerox copies exist experimentally receives a natural explanation in this model.

These inequivalent fibre bundles have already been discussed in the literature as non-Abelian Dirac 'magnetic monopoles'. Magnetic monopoles in electromagnetism have been shown to arise as topologically distinct versions of a fibre bundle with a U_1 gauge group by Wu and Yang (1975). In fact Wu and Yang went on to discuss non-Abelian Dirac 'magnetic monopoles' which Ezawa and Tze (1976) have shown to be characterised by $\pi_1(G)$ where G is the global gauge group and π_1 the first homotopy group. It is clear that these non-Abelian 'magnetic monopoles' do not have an electromagnetic magnetic charge the way the usual Dirac (1931, 1948) magnetic monopoles or the 't Hooft (1974) Polyakov (1974) monopoles do. (The latter arise in spontaneous symmetry breaking to a residual symmetry group with a U_1 factor and are characterised by $\pi_1(H)$ where H is an isotropy subgroup of G (Ezawa and Tze 1975). In fact Mandelstam (1975) and Ezawa and Tze (1976) have identified these non-Abelian 'magnetic monopoles' and their accompanying Nielsen–Olesen (1973) vortices (also characterised by $\pi_1(G)$, Ezawa and Tze 1976) as quarks in the $G \equiv SU_3/Z_3$ case in order to model quark confinement. In the present paper we associate these non-Abelian 'magnetic monopoles' with particle families. In other words, we associate the topologically distinct fibre bundles known to be associated with a given gauge theory (especially GUT theories) with the particle generations.

How far can we go with such a model of particle families? The mass splittings between the families are the most crucial experimental data to be determined. These presumably follow from the dynamics carried out in fibre bundles with inequivalent twists. Some work on twisted fields on non-trivial topologies has been done (Isham 1978) but much remains to be developed. We will not address this and other equally pressing questions in the present paper but will take only the first small step and ask what this model would give for the number of allowed generations or families for various gauge groups. Thus we will describe the mathematics of the classification of fibre bundles in § 2 below. We then apply this to the various gauge groups of interest to high energy physics in § 3 and draw our conclusions about the allowed number of generations in § 4.

2. Classification of fibre bundles

Fibre bundles are the natural mathematical description of any gauge theory (Daniel and Viallet 1980). The gauge group becomes the structure group G (and also the typical fibre F for the case of a principal fibre bundle) and spacetime the base space B of the fibre bundle E. The gauge potential (A_μ in the U(1) case) plays the role of the connection in the fibre bundle. Locally, *but not globally*, the fibre bundle is simply the product of the principal fibre and the base space. In more detail, using the treatment by Choquet-Bruhat *et al* (1977), let π be a continuous surjective mapping $\pi: E \to B$. Then $\pi^{-1}(x)$ is called the fibre at x, also denoted by F_x, where $x \in B$. Let B be covered by a family of open sets $\{U_j: j \in J \subseteq N\}$. Then a fibre bundle must satisfy the following.

(1) Locally the bundle is homeomorphic to a product bundle. Thus $\pi^{-1}(\mathrm{U}_j)$ is homeomorphic to $\mathrm{U}_j \times F$ for all $j \in J$. The homeomorphism $\phi_j: \pi^{-1}(\mathrm{U}_j) \to \mathrm{U}_j \times F$ has the form $\phi_j(P) = (\pi(P), \phi_j^{\Delta}(P))$. Thus $\phi_j^{\Delta}|_{F_x}$ also denoted by $\phi_{j,x}^{\Delta}$ is a homeomorphism from F_x onto F.

(2) The structure of the fibre bundle is determined by what happens in the overlap region. Let $x \in U_j \cap U_k$. The homeomorphism $\phi_{k,x}^{\Delta} \circ \phi_{j,x}^{\Delta-1}: F \to F$ is an element of the structural group G for all $j, k \in J$. If G has only one element the bundle is trivial. Also if the cover $\{U_j\}$ has only one element the bundle is trivial.

(3) The induced mapping $g_{jk}: U_j \cap U_k \to \mathrm{G}$ by $x \to g_{jk}(x) = \phi_{k,x}^{\Delta} \circ \phi_{j,x}^{\Delta-1}$ is continuous. $g_{jk}(x)$ is known as the transition function. The global topological structure of the fibre bundle is in the homeomorphisms in (2) above. Thus these mappings give the global difference between a cylinder and Möbius strip, both of which have the same base space and typical fibre.

As an example, if one is interested in static magnetic monopoles, the base space can be taken to be $\mathrm{R}_3 - \{0\}$ and the structure group U_1 (homeomorphic to S^1). Wu and Yang (1975) have shown (or one can see from the following) that topologically distinct bundles correspond to how many times the equator of S^2, which is the pullback of $\mathrm{R}^3 - \{0\}$, is wrapped around the S^1 structure group. This winding number gives the magnetic charge. Thus a particle with a magnetic charge twice the fundamental unit is topologically distinct from a particle with a magnetic charge of one, three, four, etc. times the fundamental unit. We will carry these ideas over into particle generations.

Our basic hypothesis is that the number of topologically distinct fibre bundles, all describing a given gauge theory with given global gauge group G, corresponds to the number of particle generations in the theory. We are interested in static fibre bundle configurations as in the magnetic monopole case and thus consider a principal fibre bundle with structure group G and base space $\mathrm{R}_3 - \{0\}$. We can contract the base space to S^2 without changing the topological properties of the fibre bundle. Thus we want to find the number N of topologically distinct fibre bundles with gauge group G over base space S^2.

We have the following classification theorem from Steenrod 1951: 'The equivalence classes of bundles over S^n with group G are in one-one correspondence with equivalence classes of elements of $\pi_{n-1}(\mathrm{G})$ under the operations of $\pi_0(\mathrm{G})$. Such a correspondence is provided by $\beta \to \chi(\alpha)$ where α is a generator of $\pi_n(\mathrm{S}^n)$ and $\chi: \pi_n(\mathrm{S}^n) \to \pi_{n-1}(\mathrm{G})$ is a characteristic homeomorphism of $\beta\ldots$'. We are interested in equivalence classes of bundles over S^2 from above. These are in one-one correspondence with equivalence classes of elements of $\pi_1(\mathrm{G})$ under the operations of $\pi_0(\mathrm{G})$. The homotopy group π_0 and π_1 measure connectedness and simply connectedness respectively. Thus we need $\pi_1(\mathrm{G})$ and $\pi_0(\mathrm{G})$ for various gauge groups G of interest to high energy physics. Since all the groups we will be concerned with are connected, we have $\pi_0(\mathrm{G}) = 0$, and we need only be concerned with $\pi_1(\mathrm{G})$, which will classify our fibre bundles and give us the number of particle generations.

For most purposes of model building in GUTS theories, only the Lie algebra of the gauge group is needed, since only the local group properties are required (Slansky 1981). Two groups with the same Lie algebra can have very different global properties, in particular their connectivity properties can be quite different. In the present case we need $\pi_1(G)$ so we are interested in the global properties of the group.

Now for any given Lie algebra g, there is a unique simply connected Lie group SG with this Lie algebra (the universal covering group) (Gilmore 1974). Also any given Lie group G with Lie algebra g is isomorphic to SG/D where D is one of the discrete invariant subgroups of SG. We also then have (Gilmore 1974)

$$\pi_1(G) \simeq D. \tag{1}$$

Finally, all discrete invariant subgroups D of SG are subgroups of the centre Z of SG (Gilmore 1974). The centre of a group G is the totality of elements of G which commute with all the elements of G (Šuzuki 1982).

Thus to classify the number of topologically distinct fibre bundles associated with a given gauge theory with Lie algebra g, we need to find the universal covering group with this g, and its centre. We then need to find all the subgroups D of the centre. Finally, (1) classifies the fibre bundles for a given choice of D and hence of 'gauge group' or global structure group G. For any given Lie algebra, we typically will have a variety of possible Lie groups G all with the *same* local GUTS physics but with different connectivity properties and hence with different numbers of associated topologically distinct fibre bundles. This would imply, for our basic hypothesis, different possible numbers of allowed particle generations. We will apply this to various Lie algebras of interest to high energy physics in the next section.

3. Application to various gauge groups

The effective gauge group which describes physics depends upon the energy scale. The Lie algebra of $U_1 \times SU_2 \times SU_3^{(c)}$ works very well at low energies, whereas the Lie algebra of something like SU_5 or SO_{10} is very likely required at the very high energy scale where the three unified interactions have the same coupling constant. We expect that the number of particle generations depends upon the number of topologically distinct fibre bundles at the grand unified (GUTS) level and not on the low energy physics. Let us see first what happens if we look at the fibre bundles with structure groups associated with the low energy effective Lie algebra, since this is instructive for the GUTS level. We then turn to the more relevant GUTS level.

At 'low' energy (energies where the weak and electromagnetic interactions are unified), we need a global group with the same Lie algebra as $U_1 \times SU_2 \times SU_3^{(c)}$. The universal covering group of U_1 is R. SU_2 and SU_3 are their own universal covering groups with centres Z_2 (elements $+1$, -1) and Z_3 (elements 1, $\exp(\pm i2\pi/3)$) respectively. $SU_2/Z_2 \simeq SO_3$ so that a partial listing of groups with the same Lie algebra as $U_1 \times SU_2 \times SU_3^{(c)}$ is (Michel 1964) $\{R \text{ or } U_1\} \times \{SU_2 \text{ or } SO_3\} \times \{SU_3 \text{ or } SU_3/Z_3\}$ (any of the eight possible combinations), $U_2 \times \{SU_3 \text{ or } SU_3/Z_3\}$, and $\{SU_2 \text{ or } SO_3\} \times U_3$. Now, since we want fermion representations, we want SU_2 to be present. Also, at this level of unification, hypercharge is not quantised so R rather than U_1 is likely to be present. Slansky (1982) argues that the global group should have a U_3 factor because of the connection between triality of colour and electric charge. This connection, however, is very likely an artefact of having a GUTS group in the background. At this level, colour confinement is put in by hand, and thus so is the above connection between triality of colour and electric charge. These considerations narrow our possibilities to $R \times SU_2 \times \{SU_3 \text{ or } SU_3/Z_3\}$.

To go further we need the following theorem (Hu 1959): 'Let X and Y be two given spaces and $x_0 \in X$, $y_0 \in Y$ be given points. Consider the product space $Z = X \times Y$ and the point $z_0 = (x_0, y_0)$ in Z. Then for every $n > 0$, we have $\pi_n(Z, z_0) \approx \pi_n(X, x_0) \times \pi_n(Y, y_0)$ where $\times$ denotes the direct product and $\approx$ denotes an isomorphism'. Applying this to the above groups gives

$$\pi_1[\mathrm{R} \times \mathrm{SU}_2 \times \mathrm{SU}_3] = 0 \tag{2}$$

$$\pi_1[\mathrm{R} \times \mathrm{SU}_2 \times \mathrm{SU}_3/Z_3] = \pi_1[\mathrm{SU}_3/Z_3] = Z_3 \tag{3}$$

where we have used (1). Thus either one-particle generation is present or three generations are present depending on whether SU_3 or SU_3/Z_3 is taken to be the effective gauge group in that sector. If we want to associate the number of generations with the number of topologically distinct fibre bundles, then the global gauge group $\mathrm{R} \times \mathrm{SU}_2 \times \mathrm{SU}_3/Z_3$ is clearly what we want, since it gives three generations. (Also note that if either U_1 or U_3 were present we would get a ZZ factor in π_1, where ZZ is the additive group of integers. This just corresponds to the usual infinite number of topologically distinct magnetic monopoles.)

We now run into trouble. SU_3/Z_3 has zero triality and does not allow representations which could describe coloured quarks. We must have $\mathrm{SU}_3^{(c)}$ itself in that sector to get the correct physics. Thus we are forced to use $\mathrm{R} \times \mathrm{SU}_2 \times \mathrm{SU}_3$ as the global gauge group and we get only one generation. Clearly if our idea of associating topologically distinct fibre bundles with particle generations is to work, it must be applied at the GUTS level. This is what we would expect. Let us now turn to these GUTS groups.

If we insist on a GUTS theory with (1) a simple Lie group, (2) complex representations, and (3) rank ≥ 4 we can only have groups with the same Lie algebra as SU_n for $n \geq 5$, SO_{4l+2} for $l \geq 2$, and E_6 (Tye 1982). We shall concentrate on SU_5, SO_{10}, and E_6 in particular.

The centre of SU_n is the group composed of the nth roots of 1. Taking SU_5 as an example (Gilmore 1974, Wybourne 1974) the centre is

$$Z = (1, \mathrm{e}^{\mathrm{i}2\pi/5}, \mathrm{e}^{\mathrm{i}2\pi 2/5}, \mathrm{e}^{\mathrm{i}2\pi 3/5}, \mathrm{e}^{\mathrm{i}2\pi 4/5})\mathrm{I}_5 \tag{4}$$

where I_5 is the identity. This is a group under multiplication and is isomorphic to Z_5 the group of integers modulo 5. All of the subgroups of Z are

$$\begin{aligned} &(1)\ \mathrm{I}_5 \\ &(Z \text{ itself}) \cong Z_5. \end{aligned} \tag{5}$$

If we choose the global gauge group to be SU_5 we get one generation; if we choose it to be SU_5/Z_5 we get five allowed generations. Similarly for groups with the same Lie algebra as some of the other SU_n groups we have

$$\begin{aligned} &\mathrm{SU}_5\text{:}\quad 1, 5 \text{ allowed generations} \\ &\mathrm{SU}_6\text{:}\quad 1, 2, 3, 6 \text{ allowed generations} \\ &\mathrm{SU}_7\text{:}\quad 1, 7 \text{ allowed generations} \\ &\mathrm{SU}_8\text{:}\quad 1, 2, 4, 8 \text{ allowed generations} \\ &\mathrm{SU}_9\text{:}\quad 1, 3, 9, \text{ allowed generations} \end{aligned} \tag{6}$$

depending on the choice of global gauge group.

Looking at SU_5/Z_5 in more detail, can we really take this to be our global gauge group or do we run into the same kind of trouble as we did for SU_3/Z_3? SU_5/Z_5 has zero quintality as defined by Slansky (1981). Thus we must only use zero quintality irreducible representations in our model building. The two lowest dimensional irreducible representations of SU_5 with zero quintality are 24 and 75, using the tables in Slansky (1981). The branching rules for SU_5 into $SU_2\times SU_3\times U_1$ are $24=(1,1)(0)+(3,1)(0)+(2,3)(-5)+(2,\bar{3})(5)+(1,8)(0)$ and $75=(1,1)(0)+(1,3)(10)+(2,3)(-5)+(1,\bar{3})(-10)+(2,\bar{3})5+(2,\bar{6})(-5)+(2,6)(5)+(1,8)(0)+(3,8)(0)$ for the two cases. The numbers in parentheses express the SU_2 content, SU_3 content, and the value of the U_1 generator respectively. Both of these representations rather miraculously contain the $(1,1)(0)$ which the Higgs sector *must* contain if spontaneous symmetry breaking is to take place correctly. If we also use 24 or 75 for the fermions, we end up with too much left-right symmetry and also unobserved colour octets. It is not at all clear that a viable theory with the correct particle content of low energies can be constructed using these irreducible representations, but it may be possible. In any case, we certainly do not have the usual model. A better possibility may be something larger like SU_9/Z_3 giving three generations. SU_9/Z_3 has a remaining triality and it seems clear that a viable theory can be constructed. Let us turn now to the SO_{4l+2} groups.

The universal covering group of SO_{4l+2} is $Spin_{4l+2}$ which is not isomorphic with a classical group for $l\geqslant 1$ (Gilmore 1974). The centre of $Spin_{4l+2}$ is isomorphic to Z_4 (Wybourne 1974, Curtis 1971). The centre thus has elements $(1, e^{i2\pi/4}, e^{i2\pi 2/4}, e^{i2\pi 3/4})I_4$ and subgroups

$$\begin{array}{ll} (Z_4 \text{ itself}) & 4 \text{ generations} \\ (1, e^{i2\pi 2/4})I_4 & 2 \text{ generations} \\ (1)I_4 & 1 \text{ generation.} \end{array} \tag{7}$$

Thus any group with the same Lie algebra as SO_{4l+2} for $l\geqslant 2$ has

$$SO_{4l+2}\colon\quad 1, 2, 4 \text{ allowed generations} \tag{8}$$

for all $l\geqslant 2$, depending on whether the choice of global gauge group is $Spin_{4l+2}$, SO_{4l+2}, or $Spin_{4l+2}/Z_4$ respectively. $\pi_1(Spin_{4l+2}/Z_4)\cong Z_4$ allowing 4 generations, for example.

If we look at the global group $Spin_{4l+2}/Z_4$ in more detail for $l=2$ using the tables in Slansky (1981), we find that we want irreducible representations of congruency class (00) which also contain $1(0)$ for branching to $SU_5\times U_1$ so that the Higgs sector will work. Irreducible representations of dimension 45 and 210 have these properties. 45, for example, has $45=1(0)+10(4)+\overline{10}(-4)+24(0)$ for branching into $SU_5\times U_1$. Of course, the breaking may go to $SU_2\times SU_2\times SU_4$ in which case $54=(1,1,1)+(3,3,1)+(1,1,20')+(2,2,6)$ or 210 are required. These also are of congruency class (00) so that they are irreducible representations of $Spin_{10}/Z_4$ as we require. Again it is not clear whether a completely viable theory can be built out of these representations.

The simply connected covering group corresponding to the Lie algebra of E_6 has centre Z_3 (Curtis 1971). This has subgroups I_3 and Z_3 itself. Thus any global group with the same Lie algebra as E_6 has

$$E_6\colon\quad 1, 3 \text{ allowed generations,} \tag{9}$$

depending on the choice of global group.

We are interested in irreducible representations of E_6/Z_3 if we are to get three particle generations. These have zero triality. The lowest dimensional representation with zero triality is 78. The branching of this into $SO_{10} \times U_1$ is (Slansky 1981) $78 = 1(0) + 45(0) + 16(-3) + \overline{16}(3)$ which rather miraculously again contains the 1(0) required for the Higgs and spontaneous symmetry breaking. This representation is so large that a viable theory can very likely be built in terms of it.

4. Discussion and conclusion

The basic idea that the number of generations of elementary particles in a gauge theory characterised by a given Lie algebra is the same as the number of topologically distinct principal fibre bundles with a structure group having the same Lie algebra and $R^3 - \{0\}$ as base space is very attractive philosophically. It requires nothing more than the usual model with an additional specification of the global gauge group which goes along with the usual Lie algebra. The topology provides a distinction between two different generations as having a different global structure or 'twist' to their fibre bundles. This provides a kind of generation quantum number but one that obviously can be broken since we have reactions like $\Lambda \to P^+ + \pi^-$ where an s quark belonging to one generation becomes a d quark belonging to another.

Equations (6), (8), and (9) give the number of allowed generations for various GUT theories. We have at most 3 generations in groups with the same Lie algebra as E_6, at most 4 generations in groups with the same Lie algebra as SO_{4l+2} with $l \geq 2$, and at most n generations in groups with the same Lie algebra as SU_n.

Acknowledgments

I would like to thank Derek Pursey and Thomas Rizzo for helpful discussions.

References

Choquet-Bruhat Y, DeWitt-Morette C and Dillard-Bleick M 1977 *Analysis Manifolds and Physics* (Amsterdam: North-Holland)
Curtis C W 1971 in *Finite Simple Groups* ed M Powell and G Higman (London: Academic) p 135
Daniel M and Viallet 1980 *Rev. Mod. Phys.* **52** 175
Dirac P A M 1931 *Proc. R. Soc.* A **133** 60
—— 1948 *Phys. Rev.* **74** 817
Ezawa Z F and Tze H C 1975 *Nucl. Phys.* B **100** 1
—— 1976 *J. Math. Phys.* **17** 2228
Gilmore R 1974 *Lie Groups Lie Algebras and Some of Their Applications* (New York: Wiley)
Glashow S 1980 *Rev. Mod. Phys.* **52** 539
Greenberg O, Mohapatra R and Yasué M 1983 *Phys. Rev. Lett.* **51** 1737
Hu S-T 1959 *Homotopy Theory* (New York: Academic) p 144
Isham C J 1978 *Proc. R. Soc.* **362** 383
Mandelstam S 1975 *Phys. Lett.* **53B** 476
Michel L 1964 in *Group Theoretical Concepts and Methods in Elementary Particle Physics* ed F Gürsey (New York: Gordon and Breach)
Nielsen H B and Olesen P 1973 *Nucl. Phys.* B **61** 45
Polyakov A M 1974 *Zh. Eksp. Teor. Fiz. Pis. Red.* **20** 430 (*JETP Lett.* **20** 194)
Ross G G 1984 *Grand Unified Theories* (Menlo Park: Benjamin/Cummings)
Salam A 1980 *Rev. Mod. Phys.* **52** 525
Slansky R 1981 *Phys. Rep.* **79** 1

Steenrod N 1951 *The Topology of Fibre Bundles* (Princeton: Princeton University Press)
Suzuki M 1982 *Group Theory I* (Berlin: Springer)
t'Hooft G 1974 *Nucl. Phys.* B **79** 276
Tye S-H 1982 *Preprint CLNS-82/527* Cornell University
Weinberg S 1980 *Rev. Mod. Phys.* **52** 515
Wu T and Yang C 1975 *Phys. Rev.* D **12** 3845
Wybourne B G 1974 *Classical Groups for Physicists* (New York: Wiley)